Time reversal invariance and the arrow of time in classical electrodynamics

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Time reversal invariance, T, is well known to hold in quantum electrodynamics. However, it has caused difficulties in classical electrodynamics (CED). These are shown to be conceptual misunderstandings. When corrected: (1) the classical equations of motion of a charge are T invariant despite the explicit occurrence of retarded fields in the equations of motion. (2) Advanced fields violate causality and occur neither in nature nor in the CED that describes it. (3) The nonexistence of advanced fields in nature implies an "arrow of time" for electromagnetic radiation: radiation emission is an overall dissipative phenomenon. This electromagnetic "arrow of time" does not contradict time reversal invariance.

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Proofs of time reversal invariance, T, of Maxwell's equations have been known for a long time both in quantum electrodynamics [1] and in classical electrodynamics (CED) [2]. The concern here, however, is not with Maxwell's equations but with the classical equations of motion of a charged particle. In QED, these are field equations (Dirac or Klein-Gordon) but in CED, they are point particle equations and depend explicitly on *retarded* interactions. Therefore, if retarded fields transform into advanced fields under T (as is commonly believed) [2], CED is not time reversal invariant. This lack of invariance claim seems to be confirmed by the occurrence of both, a second and a third time derivative in the Lorentz-Abraham-Dirac equations. But how can the classical limit of a T invariant QED yield a non-T invariant CED?

I shall first explain why retarded fields remain retarded under time reversal. Then I shall prove that T *does* hold for CED. Finally, I shall show that the absence of advanced fields is at least in part responsible for an "arrow of time" in CED.

Time reversal, T, is defined for an arbitrary point (t,x) by the map

$$T: \quad t \to t' = -t, \quad \boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x}. \tag{1}$$

A classical point charge, q, moves relativistically on a world line, $z(\tau)$, with four-velocity $v^\mu(\tau)=dz^\mu(\tau)/d\tau$, the proper time being defined by $d\tau=dz^0/\gamma$, $\gamma=(1-v^2)^{-1/2}$. The four-velocity $v^\mu(\tau)=(\gamma(\tau),\gamma(\tau)v(\tau))$ transforms under T in Minkowski space as follows: the three-velocity according to (1) transforms as $v(\tau)\to v'(\tau')=-v(\tau')$, and $\gamma(\tau)\to\gamma(\tau')=dz'^0(\tau')/d\tau'=dz^0(\tau)/d\tau=\gamma(\tau)\geqslant 1$. Therefore, $v'^\mu(\tau')=(\gamma(\tau'),-\gamma(\tau')v(\tau'))$. When one chooses the metric $(-1,+1,+1,+1)$, a point x^μ in Minkowski space M transforms into M' under T as $x^\mu\to x_\mu$, and the velocity as $v^\mu(\tau)\to v'^\mu(\tau')=-v_\mu(\tau')$.

The future is characterized in M by the increase of the proper time, τ, and in M' by the increase of τ'. The tangent vectors to the original world line and to the time reversed world line (the respective velocity four-vectors) point into opposite future light cones in M and M'. If a charge is moved by an external force from its initial conditions at τ_i to its final state at τ_f, the time reversed charge is moved by the time reversed force from the initial conditions at $\tau_i'=\tau_f$ to its final state at $\tau_f'=\tau_i$. With time inversion of the external force, the same world line is traversed in the opposite time direction. Thus, for point charges, time reversal means *motion reversal*. It is *not* a reflection in Minkowski space by the hyperplane $t=0$.

The Newtonian equations of motion are well known to be T invariant nonrelativistically as well as relativistically. But the charged particle equations require an additional term because of radiation emission, and that term depends on the *retarded* field. This fact can possibly violate T invariance. I shall show that it does not.

The charge q produces a four-current (using Gaussian units and $c=1$)

$$j^\mu(\boldsymbol{x},t) = q\int \delta_4(x-z)v^\mu(\tau)d\tau. \tag{2}$$

The current fourvector, $j^\mu=(\rho,j)$, transforms like the four-velocity,

$$T: \quad j^\mu(t,\boldsymbol{x}) \to -j_\mu(t',\boldsymbol{x}). \tag{3}$$

The time reversal transformations of the electromagnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, produced by a moving charge q are easiest derived from their potentials, $A^\mu(\phi,\boldsymbol{A})$ using $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$. In the Lorenz gauge, $\partial_\mu A^\mu=0$, the Maxwell equations reduce to $\Box A^\mu=-j^\mu$ Thus, if the four-potentials transform like the four-current, they yield

$$A^\mu(x) \to -A_\mu(x'), \tag{4}$$

as suggested by (3). The Maxwell equations would then be T invariant.

At a field point, P, with $x=(t,\mathbf{x})$, the fields are the result of the *retarded* radiation from the charge q at point Q on the world line at $z(\tau)$. With the above metric, the vector to P at x from Q at $z(\tau)$, $x-z(\tau)$, lies on the future light cone from z. When projected onto the spacelike hyperplane normal to v^μ, it becomes a spacelike four vector of unit vector u^μ and of

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magnitude $\rho(x,\tau)=u^{\mu}(\tau)(x-z(\tau))_{\mu}>0$ The retarded potential A^{μ}_{ret} at a space point x^{μ} due to a charge q traversing the world line $z^{\mu}(\tau)$ in the positive τ direction is therefore given by [2]

$$A^{\mu}_{\text{ret}}(x,\tau)=qv^{\mu}(\tau)/\rho(x,\tau). \tag{5}$$

Under T, the future light cone relative to τ (retarded radiation) becomes the light cone from Q into *its* future which is now relative to τ' (retarded radiation again!). Relative to τ', the velocity four vector, $v'^{\mu}(\tau')$, determines which light cone is the future light cone. Therefore, the transformed A^{μ} is *also* retarded. And since the distance $\rho(x,\tau)\to\rho(x',\tau')$ is invariant (and remains positive), A^{μ} transforms like v^{μ} and *remains retarded*,

$$T\colon\quad A^{\mu}_{\text{ret}}(x,\tau)\to A'^{\mu}_{\text{ret}}(x',\tau')=-A^{\text{ret}}_{\mu}(x',\tau'). \tag{6}$$

The point is that "retarded" and "advanced" always refer to the time direction in which the charges move that emit those fields. Radiation is *always* emitted into the *future* light cone and is thus always retarded. That holds also in the time-reversed world. Radiation into the past violates causality. Advanced radiation, therefore, does not exist in nature.

Equation (6) confirms (4) and the T invariance of Maxwell's equations. From (6) also follows immediately,

$$T\colon\quad F^{\mu\nu}_{\text{ret}}(x)\to -F^{\text{ret}}_{\mu\nu}(x'). \tag{7}$$

Now consider the equations of motion of a charge q. In electrodynamics, the relativistic Newtonian force equations, $mdv^{\mu}/d\tau=G^{\mu}$, cannot be maintained, because *they do not account for the emitted radiation*.

The correct equations of motion for charged particles were first derived by Lorentz and Abraham in the beginning of the 20th century (before 1905!), and later, covariantly, by Dirac [3].

The derivation of the LAD equations by Dirac starts with

$$m_{\text{bare}}dv^{\mu}/d\tau=G^{\mu}+F^{\mu}_{\text{ret}}, \tag{8}$$

where $F^{\mu}_{\text{ret}}=qF^{\mu\alpha}_{\text{ret}}v_{\alpha}$ is the particles own field, and G^{μ} is the applied external force (which need not be electromagnetic). The identity

$$F^{\mu\nu}_{\text{ret}}=(1/2)(F^{\mu\nu}_{\text{ret}}+F^{\mu\nu}_{\text{adv}})+(1/2)(F^{\mu\nu}_{\text{ret}}-F^{\mu\nu}_{\text{adv}})=F^{\mu\nu}_{+}+F^{\mu\nu}_{-} \tag{9}$$

separates this retarded field into a covariant Coulomb field and a radiation field. Note that this is a separation by time symmetry not by time inversion symmetry. The first term, $F^{\mu\nu}_{+}$, gives the electromagnetic correction to the inertial term, the second term, $F^{\mu\nu}_{-}$, gives the self-interaction force, Γ^{μ}, due to the remaining field. Thus,

$$qF_{+}{}^{\mu\alpha}v_{\alpha}=-m_{\text{elm}}dv^{\mu}/d\tau,\quad qF_{-}{}^{\mu\alpha}v_{\alpha}=\Gamma^{\mu}. \tag{10}$$

The calculation requires that one starts with the self-field on a finite radius cylinder that surrounds the world line. Then one takes the limit to zero radius [3,2]. Note that the advanced fields occur in (9) only as a mathematical convenience. They do not enter as physical quantities. In fact, Dirac's result (12) can also be derived without that artifice [4].

After combining the mass terms using $m=m_{\text{bare}}+m_{\text{elm}}$, the equations are

$$mdv^{\mu}/d\tau=G^{\mu}+\Gamma^{\mu}. \tag{11}$$

The self-force includes the radiation reaction force (the negative of the rate of energy and momentum emission of radiation.)

The term Γ^{μ} in (11) has a long history. Using $\tau_0=(2/3)q^2/m$, Dirac obtained the expression

$$\Gamma^{\mu}=m\tau_0[d^2v/d\tau^2-v^{\mu}(dv^{\alpha}/d\tau)(dv_{\alpha}/d\tau)] \tag{12}$$

by an expansion. *That expansion is not T invariant*. The corresponding Eq. (11) with Γ^{μ} in (12) (known as the LAD equation) is therefore also *not* T invariant. The expansion must be carried out separately before and after time reversal of Γ^{μ}. It is specific to each time direction.

But (12) had to be modified because it leads to unphysical solutions. After many attempts at that, the correct expression for Γ^{μ} was first characterized by Spohn [5,6]. Satisfying Spohn's criteria of being in the critical manifold of solutions, the new self-force is

$$\Gamma^{\mu}(\tau)=m\tau_0(\eta^{\mu\nu}+v^{\mu}v^{\nu})dG_{\nu}/d\tau, \tag{13}$$

where $\eta^{\mu\nu}$ is the metric tensor and $m\tau_0(2/3)q^2$. Equations (11) and (13) comprise the Landau-Lifshitz equation [7]. Note that (12) and (13) are equal up to higher orders of $\tau_0 d/d\tau$ which are outside the validity limits of classical physics.

Two comments on the self-interaction term Γ^{μ}: (1) The self-interaction term is *necessary* because it includes the rate of energy-momentum emission, $dP^{\mu}_{\text{elm}}/d\tau$, that cannot be accounted for by the Newtonian equation [Eq. (11) without Γ^{μ}]. (2) The term Γ^{μ} is also *sufficient* in that higher orders of $\tau_0 d/d\tau$ involve time intervals too small for the validity of classical physics. Thus, for example, Γ^{μ} contains the term

$$\begin{aligned}\tau_0 v^{\mu}v^{\alpha}dG_{\alpha}/d\tau&=m\tau_0 v^{\mu}v^{\alpha}d^2v_a/d\tau^2\\&=-m\tau_0 v^{\mu}(dv^{\alpha}/d\tau)(dv_a/d\tau).\end{aligned} \tag{14}$$

The first equality is valid only to first order in τ_0. Expression (14) is exactly the rate at which energy-momentum of radiation is lost by the accelerated charge (the relativistic generalization of the Larmor formula). It occurs also in Dirac's Γ^{μ}, Eq. (12).

Turning to misconceptions on time reversal, the difficulty in the previous and incorrect view of time reversal arose from the last term in Eq. (9), $F^{\mu\nu}_{-}$. In that view, retarded fields become advanced fields under T [2]. That changes the sign of Γ^{μ} under time reversal in the equation of motion (11) while the other terms keep their signs. The result is lack of T invariance. But when retarded fields remain retarded, and advanced fields remain advanced, the equations of motion (11) remain invariant under T.

Explicitly, since a special case of the external force, G^{μ} is an external electromagnetic force, $G^{\mu}=qF^{\mu\alpha}_{\text{ext}}v_a$, one sees that time reversal gives

$$T: \quad G^{\mu}(\tau) \rightarrow G'^{\mu}(\tau') = G_{\mu}(\tau'). \tag{15}$$

The equations of motion (11) with Γ^{μ} given by (10) or (13) therefore transform as

$$T: \quad md^2x^{\mu}/d\tau^2 = G^{\mu} + \Gamma^{\mu} \rightarrow md^2x'^{\mu}(\tau')/d\tau'^2$$
$$= G'^{\mu}(\tau') + \Gamma'^{\mu}(\tau')$$

or

$$md^2x_{\mu}(\tau')/d\tau'^2 = G_{\mu}(\tau') + \Gamma_{\mu}(\tau'). \tag{16}$$

Therefore, the equations of motion are T invariant.

Finally, I must return to the issue of radiation dissipation. The typical emission of radiation by moving charges is dissipative in the sense that the energy and momentum of radiation absorbed from other sources is negligibly small compared to the radiation emitted. Only two alternatives could weaken that dissipation: reabsorption of the emitted radiation, or arrival and absorption of advanced radiation. The latter is excluded because it would have to come from sources in the future going in the negative time direction and arriving at the particle on a future light cone. This violates causality. The reabsorption of the charge's own radiation is possible by suitable reflection but is always smaller than the emitted radiation. The limiting case in which the emitted radiation is fully reflected back to the moving charges and is fully reabsorbed is the case of hyperbolic motion. A perfectly reflecting cylinder concentric to a uniformly accelerated particle beam reflects the emitted radiation back onto the particles resulting in complete reabsorption. With that exception, radiation emission is dissipative. It follows that there exists an arrow of time of electromagnetic radiation. And this holds despite T invariance.

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